

Workshop: Particle on a ring

Quantum aspects of physical chemistry

<http://quantum.bu.edu/courses/PLTL/5/5.pdf>
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A particle of mass m moving on a ring of radius r in the $x y$ plane ($\theta = \pi/2$, $0 \leq \phi \leq 2\pi$) is an important model quantum system. It also provides nice examples of working with operators, the properties of their eigenfunctions and eigenvalues, and time dependence of wave functions.

■ Angular momentum

1. The operator for the angular momentum of a particle moving on a ring in the $x y$ plane ($\theta = \pi/2$, $0 \leq \phi \leq 2\pi$) is $\ell_z = -i \hbar d / d\phi$. Show that this operator is hermitian, by showing that

$$\int_0^{2\pi} f(\phi)^* \ell_z g(\phi) d\phi = \int_0^{2\pi} \{\ell_z f(\phi)\}^* g(\phi) d\phi,$$

where f and g are smooth, single-valued functions of ϕ .

2. Show that $\Phi_j(\phi) = e^{ij\phi} / \sqrt{2\pi}$, where j is an integer (including zero and negative values), is normalized over the ring, $0 \leq \phi \leq 2\pi$.

3. Show that $\Phi_j(\phi)$ is an eigenfunction of angular momentum and find the expression for the eigenvalue.

4. Show, by explicit integration, that different $\Phi_j(\phi)$ are orthogonal. That is, show that $\int_0^{2\pi} \Phi_j(\phi)^* \Phi_k(\phi) d\phi = \delta_{jk}$. where

$$\delta_{jk} = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases}$$

is called the Kronecker delta.

5. Show, without doing explicit integration, that wave functions $\Phi_j(\phi)$ with different values of j must be orthogonal. That is, show that $\int_0^{2\pi} \Phi_j(\phi)^* \Phi_k(\phi) d\phi = \delta_{jk}$.

■ Kinetic energy

6. Show that if an operator, q , is hermitian, then the operator $q^2 = q q$ is also hermitian.

7. The operator for the kinetic energy of a particle of mass m moving on a ring in the $x y$ plane is $\ell_z^2 / 2I = -(\hbar^2 / 2I) d^2 / d\phi^2$, where $I = m r^2$ is the moment of inertia and r is the radius of the ring. Show that this operator is hermitian.

8. Show that $\Phi_j(\phi)$ is an eigenfunction of kinetic energy and find the expression for the eigenvalue.

9. In classical physics, kinetic energy of a particle on a ring is $\ell_z^2 / (2I)$. Reasoning by analogy, what values of angular momentum can a particle on a ring with moment of inertia I have?

10. Discuss whether you can use the hermiticity of the kinetic energy operator to predict the value of $\int_0^{2\pi} \Phi_{-|j|}(\phi)^* \Phi_{|j|}(\phi) d\phi$.

■ Average values

11. Show that if the wave function of a system is an eigenfunction of an operator q , then the average value $\langle q \rangle$ is equal to the eigenvalue of q .

12. Obtain an expression for the average value of ℓ_z , for the wave function $\Phi_j(\phi)$.

13. Obtain an expression for the average value of ℓ_z^2 , for the wave function $\Phi_j(\phi)$.

14. Obtain an expression for the rms uncertainty, $\delta\ell_z$, for the wave function $\Phi_j(\phi)$.

15. Discuss whether your result for $\delta\ell_z$ for the wave function $\Phi_j(\phi)$, makes sense.

■ Combination wave functions

16. Construct a normalized wave function, $\Phi_{c,l,j}(\phi)$, proportional to $\Phi_{|j|}(\phi) + \Phi_{-|j|}(\phi)$.
17. Show whether $\Phi_{c,l,j}(\phi)$ is an eigenfunction of angular momentum and, if so, find the expression for its value.
18. Show whether $\Phi_{c,l,j}(\phi)$ is an eigenfunction of kinetic energy and, if so, find the expression for its value.
19. Discuss whether you can use the hermiticity of the kinetic energy operator to predict the value of $\int_0^{2\pi} \Phi_{c,l,k}(\phi)^* \Phi_{c,l,j}(\phi) d\phi$.

■ Motion on the ring

A time-independent wave function, $\Phi_j(\phi)$, with energy E_j , has a corresponding time-dependent wave function $\Psi_j(x, t) = \Phi_j(x) e^{-iE_j t/\hbar}$.

20. Write down the explicit expression for the time-dependent wave function

$$\Psi_j(\phi, t) = \Phi_j(\phi) e^{-iE_j t/\hbar}.$$

21. Construct the time-dependent wave function, $\Psi_{c,l,j}(\phi, t)$, corresponding to $\Phi_{c,l,j}(\phi)$.
22. Use the stationary phase condition to show that the wave function $\Psi_j(\phi, t)$ for $j > 0$ corresponds to a particle moving around the ring in the $x y$ plane in a counter clockwise direction.
23. Use the stationary phase condition to show that the wave function $\Psi_j(\phi, t)$ for $j < 0$ corresponds to a particle moving around the ring in the $x y$ plane in a clockwise direction.
24. Show that the wave function $\Psi_{c,l,j}(\phi, t)$ corresponds to a particle that is stationary on the ring, that is, that has no net motion around the ring.
25. Sketch the probability density corresponding to $\Psi_1(\phi, t)$. Does the probability density depend on time?
26. Sketch the probability density corresponding to $\Psi_{-1}(\phi, t)$. Does the probability density depend on time?
27. Sketch the probability density corresponding to $\Psi_{c,1}(\phi, t)$. Does the probability density depend on time?
28. Discuss your answers to questions 20–22 by comparing the probability densities of $\Phi_1(\phi, t)$ and $\Phi_{c,1}(\phi, t)$.