

Logarithms Tutorial for Chemistry Students

1 Logarithms

1.1 What is a logarithm?

Logarithms are the mathematical function that is used to represent the number (y) to which a base integer (a) is raised in order to get the number x :

$$x = a^y,$$

where $y = \log_a(x)$. Most of you are familiar with the standard base-10 logarithm:

$$y = \log_{10}(x),$$

where $x = 10^y$. A logarithm for which the base is not specified ($y = \log x$) is always considered to be a base-10 logarithm.

1.2 Easy Logarithms

The simplest logarithms to evaluate, which most of you will be able to determine by inspection, are those where y is an integer value. Take the power of 10's, for example:

$\log_{10}(10) = 1$	$10^1 = 10$
$\log_{10}(100) = 2$	$10^2 = 100$
$\log_{10}(1000) = 3$	$10^3 = 1000$
$\log_{10}(10000) = 4$	$10^4 = 10000$
$\log_{10}(1) = 0$	$10^0 = 1$
$\log_{10}(0.1) = -1$	$10^{-1} = 0.1$
$\log_{10}(0.01) = -2$	$10^{-2} = 0.01$
$\log_{10}(0.001) = -3$	$10^{-3} = 0.001$
$\log_{10}(0.0001) = -4$	$10^{-4} = 0.0001$

1.3 Rules of Manipulating Logarithms

There are four main algebraic rules used to manipulate logarithms:

Rule 1: Product Rule

$$\log_a uv = \log_a u + \log_a v$$

Rule 2: Quotient Rule

$$\log_a \frac{u}{v} = \log_a u - \log_a v$$

Rule 3: Power Rule

$$\log_a u^v = v \log_a u$$

Caution! The most common errors come from students mistakenly using two **completely fictitious** rules (there are no rules that even resemble these): $\log_a(u + v) \neq \log_a u + \log_a v$ (logarithm of a sum) and $\log_b(u - v) \neq \log_b u - \log_b v$ (logarithm of a difference).

The practical implication of these rules, as we will see in the chapters dealing with thermodynamics, equilibrium, and kinetics, is that we will be able to simplify complex algebraic expressions — easily.

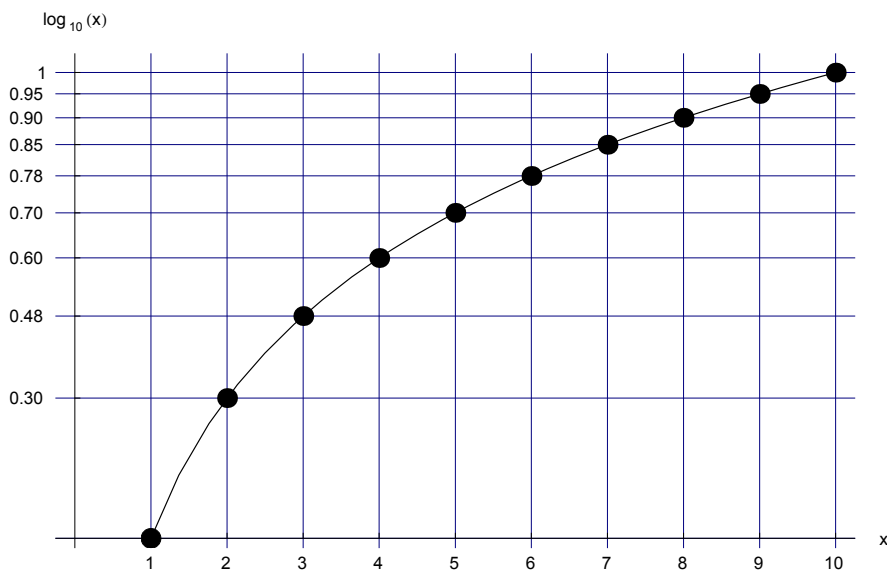
1.4 Approximating Numerical Logarithms

In order to approximate the numerical values of non-trivial base-10 logarithms we will need (a) a good understanding of the rules for manipulating logarithms and (b) the values of $\log 2$ and $\log 3$, which are 0.30 and 0.48, respectively.

Using these values and the rules we learned above, we can easily construct a table for the log values of integers between 1 and 10:

x	log x	Justification
1	0.00	By definition
2	0.30	Given
3	0.48	Given
4	0.60	$\log 4 = \log 2^2 = 2 \log 2 = 2(0.3)$
5	0.70	$\log 10 = \log 2 \cdot 5 = \log 2 + \log 5$
6	0.78	$\log 6 = \log 2 \cdot 3 = \log 2 + \log 3$
7	0.84	Estimated as $0.5(\log 6 + \log 8)$
8	0.90	$\log 8 = \log 2^3 = 3 \log 2 = 3(0.3)$
9	0.96	$\log 9 = \log 3^2 = 2 \log 3 = 2(0.48)$
10	1.00	By definition

Notice that $\log 7$ was determined using the approximation that it is the half-way point between $\log 6$ and $\log 8$. In general, as numbers become larger, the distance between their logarithms becomes smaller. Consequently, this approach should work well for large numbers. A graphical representation of this table is:



The same approach can be used for numbers larger than ten (or smaller than one). Let's outline a general approach while solving for $\log 0.0036$

1. If the number is a decimal, express the number as a whole number times 10 to a power.

$$\log 0.0036 = \log (36 \times 10^{-4})$$

2. Apply the product and power rules to separate the power of ten term and evaluate it.

$$\log (36 \times 10^{-4}) = \log 36 + (-4) \log 10 = \log 36 - 4$$

3. Express the remaining number (36) as a product of prime factors.

$$\log 36 - 4 = \log (4 \cdot 9) - 4 = \log (2^2 3^2) - 4$$

4. Apply the product and power rules to separate all of the factors and use the table for $\log 1$ to $\log 10$ to evaluate them¹.

$$\log (2^2 3^2) - 4 = 2 \log 2 + 2 \log 3 - 4 = 2(0.3) + 2(0.48) - 4 = -2.44$$

The actual value² of $\log 0.0036$ is -2.4436 .

Example: Approximating the value of $\log (2.2 \times 10^{-5})$

Following the same steps as above:

$$\begin{aligned} \log (2.2 \times 10^{-5}) &= \log (22 \times 10^{-6}) \\ &= \log 22 + (-6) \log 10 \\ &= \log (2 \cdot 11) - 6 \\ &= \log 2 + \log 11 - 6 \\ &= 0.3 + 1.04^* - 6 \\ &= -4.66 \quad (\text{Exact} = -4.658) \end{aligned}$$

*Here, $\log 11$ was computed by taking the average of $\log 10 (= 1.00)$ and $\log 12 (= 1.08)$.

Exercises:

1. Approximate, numerically, the value of the following logarithms:

(a) $\log 0.24$

(b) $\log 0.0027$

(c) $\log 0.045$

(d) $\log 810$

(e) $\log 6.3$

(f) $\log 14.7$

(g) $\log 2.8 \times 10^{-2}$

(h) $\log 1.7 \times 10^{-5}$

(i) $\log 7.3 \times 10^3$

(j) $\log 0.25^2$

(k) $\log \sqrt{1.8 \times 10^{-5}}$

(l) $\log 75^{(1/3)}$

2. Use your scientific calculator to compute the precise value of the above logarithms. If there are any significant discrepancies, try them again! This exercise can be repeated, using any random numbers, until you feel comfortable computing logarithms by hand.

¹You may run into a prime factor that is greater than 7. If that is the case, use the same approach we used to solve $\log 7$ to solve the logarithm of that prime factor.

²Computed using a Texas Instruments scientific calculator

1.5 Natural Logarithms

Natural logarithms are a specific subset of the general logarithm ($x = a^{\log_a(x)}$), where the base (a) is the number e ($= 2.718\dots$). The natural logarithm is formally defined by:

$$x = e^{\ln(x)},$$

where $\ln(x)$ ($= \log_e x$) is the ‘natural log of x ’.

To compute natural logarithms we can employ the following simple identity: $\ln(x) = 2.303 \log(x)$.

Example: Approximating the value of $\ln(2.2 \times 10^{-5})$

Following the same steps as above:

$$\begin{aligned} \ln(2.2 \times 10^{-5}) &= 2.303 \log(2.2 \times 10^{-5}) \\ &= 2.303 \times (-4.66) \\ &= -10.73 \quad (\text{Exact} = -10.72) \end{aligned}$$

*Here, $\log(2.2 \times 10^{-5})$ was taken from the exercise in the previous problem.

Exercises:

1. Approximate, numerically, the value of the following logarithms:

(a) $\ln 0.12$

(b) $\ln 0.00625$

(c) $\ln 0.064$

(d) $\ln 210$

(e) $\ln 5.5$

(f) $\ln 12.4$

(g) $\ln 3.6 \times 10^{-3}$

(h) $\ln 2.5 \times 10^{-7}$

(i) $\ln 8.3 \times 10^2$

(j) $\ln 0.18^3$

(k) $\ln \sqrt{4.9 \times 10^{-4}}$

(l) $\ln 25^{(1/4)}$

2. Use your scientific calculator to compute the precise value of the above logarithms. If there are any significant discrepancies, try them again! This exercise can be repeated, using any random numbers, until you feel comfortable computing logarithms by hand.

2 Antilogarithms

The antilogarithm, or power, function effectively undoes a logarithm. The best example of this in Chemistry is to compute the hydronium ion concentration from the pH. In this case,

$$\text{pH} = -\log[\text{H}_3\text{O}^+],$$

and the hydronium ion concentration can be found from the pH using:

$$[\text{H}_3\text{O}^+] = 10^{-\text{pH}}.$$

2.1 Approximating Base-10 Antilogs

Consider the antilog of -2.16 . The procedure for computing the power, $10^{-2.16}$, is as follows:

1. Rewrite the power as ten to the power of the difference of two numbers: a number between 0 and 1, and an integer.

$$10^{-2.15} = 10^{0.85-3}$$

2. Separate the terms using the identity $10^{a+b} = 10^a 10^b$.

$$10^{0.85-3} = 10^{0.85} \times 10^{-3}$$

3. Use the definition of a base-10 logarithm ($x = 10^y$) to determine the value of x . The easiest way to do this is to use the logarithm graph (or table) from section 5.4. In this case, $10^{0.85} \approx 7$.

$$10^{0.85} \times 10^{-3} \approx 7 \times 10^{-3}$$

Example: Approximating the value of $10^{-4.74}$

Following the same steps as above:

$$\begin{aligned} 10^{-4.74} &= 10^{0.26-5} \\ &= 10^{0.26} 10^{-5} \\ &\approx 2 \times 10^{-5} \quad (\text{Exact} = 1.8 \times 10^{-5}) \end{aligned}$$

Here, $10^{0.26}$ is approximated as 2 from the graph in section 1.4.

Exercises:

- Approximate, numerically, the value of the following antilogs to 1 significant figure:

(a) $10^{1.5}$	(b) $10^{12.3}$	(c) $10^{4.91}$
(d) $10^{-2.28}$	(e) $10^{-5.71}$	(f) $10^{-17.44}$
- Use your scientific calculator to compute the precise value of the above logarithms. If there are any significant discrepancies, try them again! This exercise can be repeated, using any random numbers, until you feel comfortable computing logarithms by hand.

3 Significant Figures involving Logarithms

The correct use of significant figures in logarithms can be a little tricky. A logarithm tells us both the power of the base (for example, 10 for base-10 logarithms) and the number multiplying the base.

For example, if we take the base-10 logarithm base 10 of $x = 1.234 \times 10^{+5}$, we get

$$\log x = \log 1.234 \times 10^{+5} = \log 1.234 + \log 10^5.$$

For the first term, $\log 1.234$, we will report the result to as many digits to the right of the decimal point as there are significant figures in the original number. Since 1.234 has four significant figures, then its logarithm will have four decimal places. For the second term, since the exponent is an integer, the result will have (essentially) an infinite number of significant figures; that means that it will not affect the final digits in the result. So our logarithm of $1.234 \times 10^{+5}$ is computed as

$$\log 1.234 + \log 10^5 = 0.0913 + 5 = 5.0913$$

Again, notice that the result of taking the logarithm of 1.234 (a number with four significant figures) is 0.0913 (a number with four decimal places).

The same rules apply in reverse to antilogarithms. For example, consider the antilog of 6.48: because the number 6.48 has two digits to the right of the decimal point, we would get an antilog with two significant figures.

$$10^{6.48} = 10^{0.48} \times 10^6 = 3.0 \times 10^6$$

If these rules seem confusing at first, that is ok – many people have struggled with them. For more details, see the article by D. E. Jones in the Journal of Chemical Education, 1971, volume 49, page 753.