Appendix 2, Approximating with logarithms

Many students are rusty on logarithms, probably because most work with them takes place behind the scenes in the circuits of electronic calculators, but I think you will find with some review you will be able to use them easily.

There are two reasons why it is important to be able to work with logarithms. First, it then becomes easy to estimate numerical results of otherwise complicated expressions. Second, you will be able to work with logarithms of *algebraic* rather than numerical quantities, which is essential to understand key aspects of chemistry.

These notes describe approximating numerical values of logarithms and antilogarithms, including how to set the number of significant figures. The first part of these notes, on approximating numerical values of logarithms, was originally prepared by George Huber, a senior teaching fellow in general chemistry at Boston University for several years.

■ Definition of logarithm

To begin, recall that the logarithm base 10 of x is the power to which 10 is raised to equal x,

$$x = 10^{\log_{10}(x)}.$$

In the following, log(…) means logarithm base 10. We also will use the natural logarithm. The natural logarithm has the base $e = 2.718\ldots$, that is, the natural logarithm of x is the power to which e is raised to equal x,

$$x = e^{\log_e(x)}.$$

In the following, ln(…) means logarithm base e.

■ Approximating numerical values of logarithms

In this method, you need to remember the two values

$$\log(2) = 0.30,$$

$$\log(3) = 0.48,$$

as well as the two rules for logs

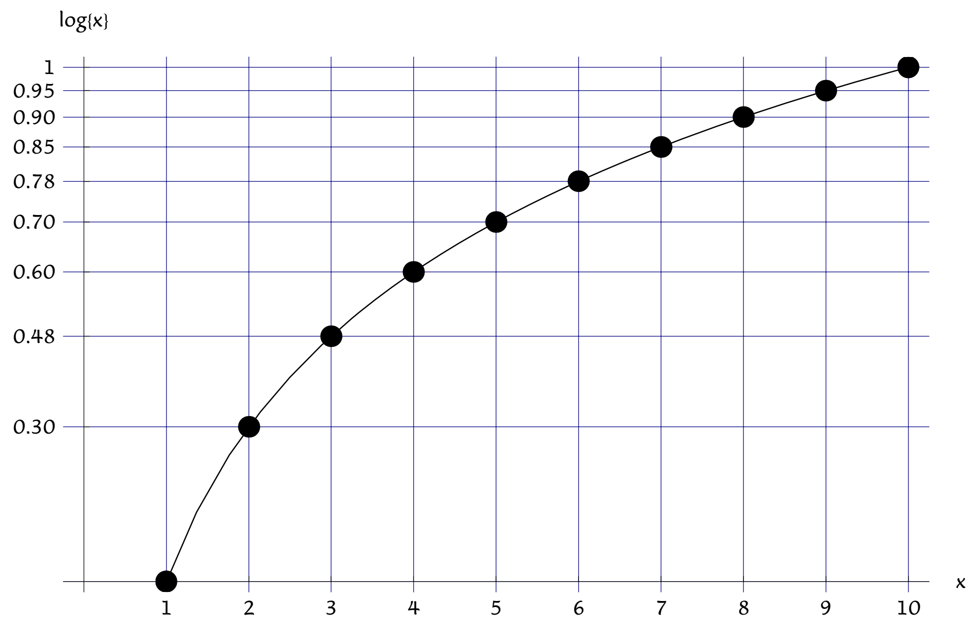
$$\log(a\,b) = \log(a) + \log(b),$$

$$\log(a)^m = m\log(a).$$

With this information you can construct the following table:

x	log(x)	comments
1	0	by definition
2	0.30	given
3	0.48	given
4	0.60	because $\log(4) = \log(2)^2 = 2\log(2)$
5	0.70	because $\log(10) = \log(2) + \log(5)$
6	0.78	because $\log(6) = \log(2) + \log(3)$
7	0.84	see below
8	0.90	because $\log(8) = 3\log(2)$
9	0.96	because $\log(9) = 2\log(3)$
10	1	by definition

Here is a graphical representation of the table.



Values of log(x) for $1 \le x \le 10$.

To get a value between any two tabulated values, you would use a linear interpolation or, if available, estimate the value using the graph.

For example to evaluate log(3.5), you would notice that 3.5 is halfway between 3 and 4, so you would make the guess that the log is also halfway between 0.48 and 0.60—this is 0.54; a more exact value is 0.544.

To get the log of a value greater then 10, first write the number in scientific notation, as a number between 1 and 10 times a power of ten, and then use the first rule above to break it into two parts, evaluate each part and then combine your two answers. For example, here is how to evaluate the log of 365.

$$
\begin{aligned}
&\log(365)\\
&\quad = \log(3.65 \times 10^2)\\
&\quad = \log(3.65) + \log(10^2)\\
&\quad = \log(3.65) + 2\\
&\quad = 0.56 + 2\\
&\quad = 2.56
\end{aligned}
$$

A more exact answer is 2.562. The value of seven was determined in this way: We can note that log(35) = log(7) + log(5), and from above we can see that the log(35) = 1.54. Thus the log(7) is given by

$$\log(7) = \log(35) - \log(5) = 1.54 - 0.70 = 0.84.$$

Use the same procedure to get the log of a number less than 1, namely first write the number in scientific notation, as a number between 1 and 10 times a power of ten. For example, express 0.365 as 3.65×10^{-1} and then evaluate its log as

$$\begin{aligned}
&\log(0.365)\\
&\quad = \log(3.65 \times 10^{-1})\\
&\quad = \log(3.65) + \log(10^{-1})\\
&\quad = \log(3.65) - 1\\
&\quad = 0.56 - 1\\
&\quad = -0.44
\end{aligned}$$

■ Calculating natural logarithms

To calculate the natural logarithm of a number, use the following conversion:

$$\ln(x) = 2.303 \log(x).$$

This conversion can be verified in the following method. Let

$$y = \log(x).$$

This means that

$$x = 10^{\log(x)} = 10^{y}.$$

Now, if we take the natural log of both sides we get

$$\ln(x) = \ln(10^{y}) = y \ln(10) = \log(x) \ln(10) = 2.303 \log(x),$$

since $\ln(10) = 2.303$. This means that the natural logarithm of a number is just 2.303 times its base 10 logarithm.

■ Evaluating powers of ten (base 10 antilogarithms)

Often we have the logarithm of a number and we would like to know the number itself. An example is in acid-base equilibrium, where we may have measured the pH and would like to know the corresponding concentration of hydronium ions, $10^{-\mathrm{pH}}$.

Here is the procedure to use to evaluate the antilog of a number. As example, let's evaluate $10^{-2.13}$. The first step is to express the log as the sum of an integer and a number between 0 and 1. In this case we can write

$$-2.13 = 0.87 - 3.$$

This tells us that

$$10^{-2.13} = 10^{0.87} \times 10^{-3}$$

We can use the general scheme we have outlined earlier to determine the antilog of any number between 0 and 1. In this case, we know that

$$\log(2^3) = 3\log(2) = 3 \times 0.30 = 0.90$$

and so the antilog of 0.87 is a bit less than $8 = 2^3$. To one significant figure, then, $10^{0.87} \approx 8$ and so

$$10^{-2.13} \approx 8 \times 10^{-3}.$$

A more exact value is 7.41×10^{-3}.

■ Evaluating powers of e (base e antilogarithms)

Many processes proceed in proportion to an exponential. The general procedure to evaluate $y = e^x$ is to evaluate the base 10 logarithm of both sides, using the value

$$\log(e) = 0.4343,$$

and then caluclate the base 10 antilogarithm. As example, let's evaluate $y = e^{-7.3}$.

$$\log(y) = \log(e^{-7.3}) = -7.3\log(e) = \; -7.3 \times 0.4343 == 3.2$$

This means that

$$y = 10^{-3.2} = 10^{0.8} \times 10^{-4} = 10^{\log(6.3)} \times 10^{-4} = 6.3 \times 10^{-4}.$$

A more exact value is 6.76×10^{-3}.

An important special case is decay with a constant half life, for which the fraction remaining after n half lives is $y = e^{n\ln(2)}$. In this case we can remember the constant

$$\ln(2)\log(e) = 0.30103$$

and so use

$$\log(y) = \log(e^{-n\ln(2)}) = -n\ln(2)\log(e) = -n \times 0.30103.$$

For example, the fraction of a population remaining after 2.3 half lives is

$$y = 10^{-2.3\times0.30103} = 10^{-0.7} = 10^{0.3} \times 10^{-1} = 2 \times 10^{-1} = 0.2$$

A more exact value is $e^{2.3\ln(2)} = 0.203$.

■ Using logarithms to evaluate powers and roots

Often we will have to evaluate non-integer powers and roots of expressions, as was the case in the last example. Here is a general way to do this using logarithms. To see how it works, let's evaluate

$$x = (17.99)^{1.445}.$$

The idea is to take the log (base 10) of both sides, and use the rule that the logarithm of a quantity raised to a power is the power times the logarithm of the quantity,

$$\log(x) = 1.445\log(17.99).$$

Then, evaluate the logarithm of the quantity and evaluate the product,

$$\log(x) = 1.445 \times 1.255 = 1.814.$$

Finally, take the antilog of both sides,

$$x = 10^{1.814} = 10^{1}\, 10^{0.814} = 10 \times 6.509 = 65.09.$$

■ Significant figures involving logarithms

In doing calculations with logarithms, we need to know how to determine the number of significant figures. Significant figures involving logarithms are a little subtle, because a logarithm tells us both the power of the base (usually 10 or e) and the number multiplying this.

The starting point is to write the number whose logarithm we wish to evaluate as a number between 1 and 10 times a power of ten. For example, we would write the number 123.4×10^3 as

$$x = 1.234 \times 10^5.$$

Then, we take the logarithm base 10,

$$\log(x) = \log(1.234) + \log(10^5) = 0.0913 + 5.$$

There are two things to notice about this result. First, in the first term we *keep as many digits to the right of the decimal point as there are significant figures in the original number*, here four. Second, there are essentially an infinite number of significant figures in the second term, since the exponent (5) is exactly known. This means that we can write

$$\log(x) = 0.0913 + 5 = 5.0913$$

In taking antilogs, we need to reverse these rules: We *keep as many digits as there are digits to the right of the decimal point of the original logarithm*. For example, the logarithm 28.0913 has four digits to the right of the decimal. We get therefore

$$x = 10^{28.0913} = 1.234 \times 10^{28}.$$

This material is discussed in an article by D. E. Jones in the *Journal of Chemical Education*, 1971, volume 49, page 753. I mention this to underscore that many people (including me) have been confused by this.