Now that we know a little about where the Schrödinger comes from, we can next see what it means. The key to beginning to make sense of the Schrödinger equation is to rewrite it in a special way,

\[
\text{curvature of } \psi \text{ at } x \propto -\text{kinetic energy at } x \times \psi \text{ at } x
\]

We will see that using this form of the Schrödinger equation we can develop a qualitative understanding of all of the key features of wavefunctions and energies of quantum systems.

- that in regions of high kinetic energy wavefunctions oscillate rapidly and so have small wavelengths;
- that in regions of low kinetic energy that wavefunctions oscillate slowly and so have large wavelengths;
- that in regions of negative (!) kinetic energy, instead of oscillating, wavefunction diverge away from zero;
- and, finally, that the observed (quantized) energies, are just those energies for which the wavefunction is prevented from diverging to infinity.

We will be able to do all of this just by making simple sketches of the form of the wavefunction required by the Schrödinger equation. It is easy to make these sketches, once we learn what the rules are, and these sketches allow us to accurately predict what wavefunctions look like without doing any calculations at all.
Kinetic energy variation with position

The first step toward understanding the Schrödinger equation is to understand how kinetic energy changes with position. The kinetic energy is determined by the total energy, $E$, and the potential energy, $V(x)$. The total energy is a constant but the potential energy in general changes with position. This means that the kinetic energy changes with position.

A nice example illustrating changing potential energy of a skateboarder traversing a parabolic track. At the top of the track the skateboarder has the greatest (gravitational) potential energy but no kinetic energy; her total energy is entirely potential energy. Once she starts to roll down the track, she picks up speed as her potential energy is converted to kinetic energy. At the bottom of the track she is moving the fastest and her potential energy is at its lowest value. She continues to roll up the other side of the track, rapidly at first but slowing down more and more, as her kinetic energy is converted back into potential energy. The potential energy reaches its maximum at the top of the other side of the track, where the skateboarder comes to a stop and so has zero kinetic energy.

We can develop this example quantitatively. We begin by expressing the shape of the track in terms of its vertical height relative to the bottom of the track. Let's make the track have a parabolic profile,

$$h(x) = x^2 \text{ meter}.$$ 

We can specify how high up the track the skateboarder begins the traverse by setting her total energy equal to her gravitational potential energy, relative to the bottom of the track, at that height. The gravitational potential energy is determined by the skateboarder's mass and by the height,

$$V(x) = m g h$$

Here $m$ is the skateboarder's mass, $g$ is the acceleration due to gravity, and $h$ is the height of the skateboarder above the bottom of the track (note, $h$ is not Planck's constant here!). Let's say the horizontal length of the track, from the starting edge across to the finishing edge, is 10 meters and that the skateboarder weighs 120 lbs. The means that height of the track is

25 Meter

and that the skateboarder's total energy is

13000. Joule

Confirm these values for the height of the track and the skateboarder's total energy.

Here is a graph of the skateboarder's potential energy, $V(x)$, along the traverse.
Gravitational potential energy $V(x)$ along a parabolic track. The horizontal line indicates the total energy $E$.

The horizontal line is her total energy, $E$. The kinetic energy at any point along the traverse is the vertical distance, $E - V(x)$, from the total energy line down to the potential energy curve. The value of the skateboarder's maximum kinetic energy, at the bottom of the traverse, is

$$13000 \text{ Joule}$$

Her initial potential energy has been converted entirely to kinetic energy there.

The speed of the skateboarder at any point along her traverse is proportional to the square root of her kinetic energy at that point.

$$v(x) = \sqrt{\frac{I(x)}{2m}}$$

Here is a graph of her speed along the traverse.

Velocity along a parabolic skateboard track, starting from rest at its top at $x = \pm 5 \text{ m}$.

Her maximum speed, at the bottom of the traverse, is

$$22 \text{ Meter per Second}$$
Use the graphs to determine the kinetic energy, potential energy and speed of the skateboarder at position \( x = -3 \) meters, having started down the track at rest from the left.

We can apply the same ideas to understand how the kinetic energy of a particle with a total energy \( E \) changes due to changing values of its potential energy, \( V(x) \).

## Effect of kinetic energy on wavefunction curvature

The Schrödinger equation is a relation between the curvature of the wavefunction, the kinetic energy of the particle, and the value of the wavefunction. To see precisely what this relation means, let's consider the Schrödinger equation in detail for the possible combinations of kinetic energy and sign of the wavefunction.

In the skateboarder example, the kinetic energy increases from zero at the start and end of the traverse to its maximum value at the midpoint. The skateboarder's kinetic energy can never be negative. In the Schrödinger equation, in contrast, the kinetic energy may be either positive or negative, according to whether the value of the potential energy is smaller or larger than the total energy. Regions of \( x \) where kinetic energy is positive or zero are called classically allowed regions, or more simply, allowed regions; regions of \( x \) where kinetic energy is negative are called classically forbidden regions, or more simply, forbidden regions. Boundaries between allowed and forbidden regions are called classical turning points.

It is quite peculiar that kinetic energy may be negative, since for an ordinary object this would mean the particle velocity would have to be imaginary, since a negative kinetic energy means \( v^2 < 0 \). The reason this is possible in quantum systems is that we are dealing with probability amplitudes rather than localized particles. The squared modulus of the probability amplitude, \( |\psi(x)|^2 \), tells us the probability per unit length of finding the particle in any particular region. To compute the kinetic energy of the particle, we need to add up its kinetic energy in each region, weighting each contribution by the probability that the particle will be in that region. The result is that the negative contributions to the kinetic energy from the forbidden regions are always more than offset by the positive contributions to the kinetic energy from the allowed regions, so that the total kinetic energy is always positive.

Forbidden regions turn out to be crucial in understanding why only certain values of the energy may occur; that is, forbidden regions are the source of energy quantization. Energy quantization always means that particle is confined to a limited region of space by classically forbidden regions.

There are four possible combinations of the sign of the kinetic energy and the sign of the wavefunction: positive kinetic energy and either positive or negative wavefunctions, and negative kinetic energy and either positive or negative wavefunctions. Let's look at these combinations separately for the case of positive and negative kinetic energy.
Allowed regions

In allowed regions, the kinetic energy is never negative. This means that the sign of the curvature of the wavefunction is always opposite to the sign of the wavefunction.

\[
\text{sign of curvature of } \psi \text{ at } x = - \times \text{sign of kinetic energy at } x \times \text{sign of } \psi \text{ at } x \\
= - \times + \times \text{sign of } \psi \text{ at } x \\
= - \times \text{sign of } \psi \text{ at } x
\]

If the wavefunction is positive, then its curvature will be negative. A negative curvature means that the slope of the wavefunction is becoming more negative (less positive); that is, the wavefunction is curving downward from positive values toward the zero line.

If the wavefunction is negative, then its curvature will be positive. A positive curvature means that the slope of the wavefunction is becoming more positive (less negative); that is, the wavefunction is curving upward from negative values toward the zero line.

The net result of this behavior is that

in classically allowed regions, the wavefunction oscillates back and forth across its zero line.

In this way we see that Schrödinger equation predicts an oscillatory probability amplitudes in allowed regions.

Since the curvature — how quickly the slope of the wavefunction is changing — is proportional to the kinetic energy, where the kinetic energy is large, the oscillations will be rapid; that is, each loop of the wavefunction (wavefunction variation between two adjacent zero values) will occupy a relatively small range of \( x \) and the maximum value of the wavefunction will never have a chance to get very large. Conversely, where the kinetic energy is small, the oscillations will be slow; that is, each loop of the wavefunction will occupy a relatively large region of \( x \) and the maximum value of the wavefunction will be able to become relatively large.

The picture that emerges is consistent with de Broglie's relation, \( \lambda = \frac{h}{p} \). When kinetic energy increases, the momentum, \( p \), gets larger and the wavelength of the wavefunction becomes smaller. When kinetic energy decreases, the momentum, \( p \), gets smaller and the wavelength of the wavefunction becomes larger. (This connection with the de Broglie behavior is no accident, for de Broglie's relation is built into the Schrödinger equation, through the equation's dependence on the position-momentum commutator, \([x, p] = i\hbar\).)
Forbidden regions

In forbidden regions, the kinetic energy is always negative. This means that the sign of the curvature of the wavefunction is always the same as the sign of the wavefunction.

\[
\text{sign of curvature of } \psi \text{ at } x = - \times \text{sign of kinetic energy at } x \times \text{sign of } \psi \text{ at } x \\
= - \times - \times \text{sign of } \psi \text{ at } x \\
= + \times \text{sign of } \psi \text{ at } x
\]

If the wavefunction is positive, then its curvature will be positive. A positive curvature means that the slope of the wavefunction is becoming more positive (less negative); that is, positive wavefunctions curve away from the zero line toward more positive values.

If the wavefunction is negative, then its curvature will be negative. A negative curvature means that the slope of the wavefunction is becoming more negative (less positive); that is, the negative wavefunctions again curve away from the zero line toward more negative values.

The net result of this behavior is that in classically forbidden regions, the wavefunction curves away from its zero line.

In this way we see that Schrödinger equation predicts that probability amplitudes tend to diverge to large, possibly infinite values in forbidden regions.

As in allowed regions, since the curvature is proportional to the kinetic energy, where the negative kinetic energy is large, the curving away from the zero line will be rapid; that is, over a relatively small distance \( x \) the wavefunction value can move relatively far from the zero line. Further, this movement away from the zero line will be amplified as the wavefunction value increases, since it multiplies the kinetic energy to determine the curvature. Conversely, where the negative kinetic energy is small, the curving away from the zero line will be slow; that is, a relatively larger distance in \( x \) will be needed for the wavefunction to change appreciably. But again, because curvature depends on the product of kinetic energy and wavefunction, as the wavefunction becomes larger, the movement away from the zero line will accelerate.

Quantization

The different behavior of wavefunctions in allowed and forbidden regions is the origin of energy quantization. To see why this is so, we need to recall the rule which associates wavefunctions with particles.
$|\psi(x)|^2 \, dx$ is probability of particle being within $dx$ of $x$

Now, the total probability — the probability of the particle being somewhere — must be unity,

$$1 = \int_{-\infty}^{\infty} |\psi(x)|^2 \, dx$$

and this so-called normalization requirement can be satisfied only if the wavefunction, $\psi(x)$, does not diverge to infinite positive or negative values. Since we know that the wavefunction curves away from the zero line in the forbidden region, we can insure that it does not grow to infinite values only by arranging for it to enter forbidden regions with just the right curvature so that it curves away from the axes exponentially slowly.

For example, say the wavefunction enters a forbidden region with a positive value but negative slope. Since its slope entering the forbidden region is downward, its value will initially continue to decrease in the forbidden region. If its curvature is too great, the decrease will slow to a stop and then increase toward infinite positive values. If its curvature is too small, the decrease will carry the function to the zero line, then to negative values, and thereafter the function will increase towards infinite negative values. Only if the curvature is just right will the function decrease toward the zero line at an ever slowing rate (since the function itself is decreasing and the curvature is proportional to the value of the function), never getting to the zero line.

The way to adjust the curvature so that exponential decay is achieved is by adjusting the total energy, $E$. The special values of the energy for which the wavefunction decays to zero in forbidden regions are precisely the possible energies of the system.

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**Summary and study problem**

We have seen that wavefunctions $\psi$ that satisfy the Schrödinger equation are those that oscillate in allowed regions and curve away from the zero line in forbidden regions. The oscillations in the allowed region are slow and with large amplitude where the kinetic energy is small, and are rapid and with small amplitude where the kinetic energy is large. The curving away from the zero line usually leads to wavefunctions that diverge to infinite values. The normalization conditions means that such wavefunctions, while mathematically acceptable, are physically unacceptable. Only at special values of the energy does the divergence in the forbidden region become an exponential decay to the zero line and so are physically acceptable wavefunctions possible. These special values of the energy are the quantized energies of the system.

Airy functions are physically acceptable solutions of the one dimensional Schrödinger equation for a particular potential energy variation, $V(x)$. Here is a plot of an Airy function (thin line) and its curvature (thick line).
Airy function (thin line) and its curvature (thick line).

Here is a more detailed view of the function and its curvature in the region $5 \leq x \leq 12$.

Use the ideas developed in these notes and the details in these plots to make a sketch of the corresponding potential energy variation, $V(x)$, and indicate on your sketch the total energy and the position of the classical turning point.